On Generalized Super-Coherent States

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Abstract

A set of operators, the so-called k-fermion operators, that interpolate between boson and fermion operators are introduced through the consideration of an algebra arising from two non-commuting quon algebras. The deformation parameters q and $1/q$ for these quon algebras are roots of unity with $q = \exp(2\pi\mathrm{i}/k)$ and $k \in \mathbf{N}\backslash\{0, 1\}$. The case $k = 2$ corresponds to fermions and the limiting case $k \to \infty$ to bosons. Generalized coherent states (connected to k-fermionic states) and super-coherent states (involving a k-fermionic sector and a purely bosonic sector) are investigated.

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1 Introduction

The interest of q-deformations for statistical physics is still very high in the community of physicists and mathematicians. In recent years, many works have been devoted to statistics of q-bosons, q-fermions and quons (see, for instance, Ref. [1] and references therein). This paper is devoted to k-fermions which are objects interpolating between fermions (corresponding to $k = 2$) and bosons (corresponding to $k \to \infty$).

The material in the present paper is organized as follows. We first discuss (in Section 2) the k-fermionic algebra Σ_q, where $q := \exp(2\pi \mathrm{i}/k)$ with $k \in \mathbf{N} \setminus \{0, 1\}$, in terms of generalized Grassmann variables. Then, we introduce (in Section 3) generalized coherent states. Finally, the notion of fractional super-coherent states is introduced (in Section 4) from a certain limit of the well-known deformed coherent states.

2 The k-fermions

2.1 The k-fermionic algebra Σ_q

We first introduce the k-fermionic algebra Σ_q. The algebra Σ_q is generated by five operators a_+, a_-, a_+^+, a_-^+ and N. We assume that N is an Hermitean operator, that a_+^+ (respectively, a_-^+) is the adjoint of a_+ (respectively, a_-) and that these operators satisfy

$$a_- a_+ - q a_+ a_- = 1 \iff a_+^+ a_-^+ - \bar{q} a_-^+ a_+^+ = 1 \tag{1a}$$

$$N a_+ - a_+ N = +a_+ \iff N a_+^+ - a_+^+ N = -a_+^+ \tag{1b}$$

$$N a_- - a_- N = -a_- \iff N a_-^+ - a_-^+ N = +a_-^+ \tag{1c}$$

$$(a_+)^k = (a_-)^k = 0 \iff (a_+^+)^k = (a_-^+)^k = 0 \tag{1d}$$

$$a_- a_+^+ = \bar{q}^{\frac{1}{2}} a_+^+ a_- \iff a_+ a_-^+ = q^{\frac{1}{2}} a_-^+ a_+ \tag{1e}$$

where the complex number

$$q := \exp\left(\frac{2\pi \mathrm{i}}{k}\right) \quad \text{with} \quad k \in \mathbf{N} \setminus \{0, 1\}$$

is a root of unity. (In Eq. (1), $\bar{q}$ stands for the complex conjugate of q.) The algebra Σ_q clearly involves two non-commuting quon algebras A_q (spanned by a_+, a_- and N) and $A_{\bar{q}}$ (spanned by a_-^+, a_+^+ and N).

In view of the defining relations (1), the operators a_+, a_-, a_+^+, a_-^+ and N act on a Fock space $\mathcal{F} := \{|n\rangle : n = 0, 1, \cdots, k-1\}$ with card $\mathcal{F} = k$. Furthermore, we chose a representation of Σ_q in the following way. The action of N is standard in the sense that

$$N|n\rangle = n|n\rangle$$

while the action of the remaining operators is given by

$$a_-|n\rangle = \left([n]_q\right)^{\frac{1}{2}} |n-1\rangle \quad \text{with} \quad a_-|0\rangle = 0$$

$$a_+^+|n\rangle = \left([n]_{\bar{q}}\right)^{\frac{1}{2}} |n-1\rangle \quad \text{with} \quad a_+^+|0\rangle = 0$$

and

$$a_+|n\rangle = \left([n+1]_q\right)^{\frac{1}{2}} |n+1\rangle \quad \text{with} \quad a_+|k-1\rangle = 0$$

$$a_-^+|n\rangle = \left([n+1]_{\bar{q}}\right)^{\frac{1}{2}} |n+1\rangle \quad \text{with} \quad a_-^+|k-1\rangle = 0$$

where the symbol $[\ \ ]_q$ is defined by

$$[X]_q := \frac{1-q^X}{1-q}$$

for any operator or number X. Thus, the operators a_- and a_+^+ behave like annihilation operators, the operators a_+ and a_-^+ like creation operators and the operator N like a number operator.

The state vector $|n\rangle$ can be written as

$$|n\rangle = \frac{(a_+)^n}{([n]_q!)^{\frac{1}{2}}}\,|0\rangle \quad \text{or} \quad |n\rangle = \frac{(a_-^+)^n}{([n]_{\bar{q}}!)^{\frac{1}{2}}}\,|0\rangle \quad \text{for} \quad n = 0, 1, \cdots, k-1$$

where, as usual, the p-deformed factorial $[n]_p$ is defined by (with $p = q$ and $\bar{q}$)

$$[n]_p! := [1]_p[2]_p \cdots [n]_p \quad \text{for} \quad n \in \mathbf{N} \setminus \{0\} \quad \text{and} \quad [0]_p! := 1$$

In the specific case $k = 2$, the algebra Σ_{-1} corresponds to ordinary fermion operators with $a_+^+ = a_-$ and $a_-^+ = a_+$ for which we have $(a_-)^2 = (a_+)^2 = 0$, a relation that reflects the Pauli exclusion principle. In the limiting case $k \to \infty$, the algebra Σ_{+1} corresponds to ordinary boson operators with $a_+^+ = a_-$ and $a_-^+ = a_+$. For k arbitrary, the algebra Σ_q corresponds to k-fermion operators a_- and a_+ (with their adjoint a_-^+ and a_+^+, respectively) that interpolate between fermion and boson operators ; the space $\mathcal{F}$ is of dimension k for the k-fermionic algebra Σ_q (i.e., two-dimensional for the fermionic algebra Σ_{-1} and infinite-dimensional for the bosonic algebra Σ_{+1}).

2.2 Grassmannian realization of Σ_q

We give here some preliminaries useful for obtaining a Grassmannian realization of the algebra Σ_q. Equation (1d) suggests that we use generalized Grassmann variables (see Refs. [2-5]) z and $\bar{z}$ such that

$$z^k = \bar{z}^k = 0 \tag{2}$$

(The particular case $k = 2$ corresponds to ordinary Grassmann variables.) We then introduce the ∂_z- and $\partial_{\bar{z}}$-derivatives via

$$\partial_z f(z) := \frac{f(qz) - f(z)}{(q-1)z}, \qquad \partial_{\bar{z}} g(\bar{z}) := \frac{g(\bar{q}\bar{z}) - g(\bar{z})}{(\bar{q}-1)\bar{z}} \tag{3}$$

where $f : z \mapsto f(z)$ and $g : \bar{z} \mapsto g(\bar{z})$ are arbitrary functions. The linear operators ∂_z and $\partial_{\bar{z}}$ satisfy

$$\partial_z z^n = [n]_q \, z^{n-1}, \qquad \partial_{\bar{z}} \bar{z}^n = [n]_{\bar{q}} \, \bar{z}^{n-1}$$

for $n = 0, 1, \cdots, k-1$. Therefore, when $f(z)$ and $g(\bar{z})$ can be developed as

$$f(z) = \sum_{n=0}^{k-1} a_n z^n, \qquad g(\bar{z}) = \sum_{n=0}^{k-1} b_n \bar{z}^n$$

where the coefficients a_n and b_n in the expansions are complex numbers, we check that

$$(\partial_z)^k f(z) = (\partial_{\bar{z}})^k g(\bar{z}) = 0$$

Consequently, we shall assume that the conditions

$$(\partial_z)^k = (\partial_{\bar{z}})^k = 0 \tag{4}$$

hold in addition to Eq. (2).

From Eqs. (2) and (4), the correspondences

$$a_- \to \partial_z, \qquad a_+ \to z, \qquad a_+^+ \to \partial_{\bar{z}}, \qquad a_-^+ \to \bar{z} \tag{5}$$

clearly provide us with a realization of Eqs. (1a) and (1d). Note that Eq. (1e) leads to

$$\partial_z \partial_{\bar{z}} = \bar{q}^{\frac{1}{2}} \, \partial_{\bar{z}} \partial_z, \qquad z\bar{z} = q^{\frac{1}{2}} \, \bar{z} z$$

in the realization based on Eq. (5).

3 Generalized coherent states

There exists several methods for introducing coherent states. We can use the action of a displacement operator on a reference state [6] or the construction of an eigenstate for an annihilation operator [7,8] or the minimisation of uncertainty relations [9]. In the case of the ordinary harmonic oscillator, the three methods lead to the same result (when the reference state is the vacuum state). Here, the situation is a little bit more intricate (as far as the equivalence of the three methods is concerned) and we chose to define the generalized coherent states or k-fermionic coherent states $|z)$ and $|\bar{z})$ as follows

$$|z) := \sum_{n=0}^{k-1} \frac{z^n}{([n]_q!)^{\frac{1}{2}}} \, |n\rangle, \qquad |\bar{z}) := \sum_{n=0}^{k-1} \frac{\bar{z}^n}{([n]_{\bar{q}}!)^{\frac{1}{2}}} \, |n\rangle$$

where z and $\bar{z}$ are generalized Grassmann variables that satisfy Eq. (2). It can be easily checked that the state vectors $|z)$ and $|\bar{z})$ are eigenvectors of the operators a_- and a_+^+, respectively. More precisely, we have

$$a_-|z) = z|z), \qquad a_+^+|\bar{z}) = \bar{z}|\bar{z})$$

The case $k = 2$ corresponds to fermionic coherent states while the limiting case $k \to \infty$ to bosonic coherent states.

We define

$$(z| := \sum_{n=0}^{k-1} \langle n| \frac{\bar{z}^n}{([n]_{\bar{q}}!)^{\frac{1}{2}}}, \qquad (\bar{z}| := \sum_{n=0}^{k-1} \langle n| \frac{z^n}{([n]_q!)^{\frac{1}{2}}}$$

Then, the 'scalar products' $(z'|z)$ and $(\bar{z}'|\bar{z})$ follow from the ordinary scalar product $\langle n'|n\rangle = \delta(n', n)$. For instance, we get

$$(z'|z) = \sum_{n=0}^{k-1} \frac{\bar{z'}^n z^n}{([n]_{\bar{q}}![n]_q!)^{\frac{1}{2}}}$$

In view of the relationship

$$[n]_{\bar{q}}! = q^{-\frac{1}{2}n(n-1)}\, [n]_q!$$

and of the property

$$\bar{z}^n z^n = q^{-\frac{1}{4}n(n-1)}\, (\bar{z}z)^n$$

we obtain the following result

$$(z|z) = \sum_{n=0}^{k-1} \frac{(\bar{z}z)^n}{[n]_q!} \tag{6}$$

Similarly, we have

$$(\bar{z}|\bar{z}) = \sum_{n=0}^{k-1} \frac{(z\bar{z})^n}{[n]_{\bar{q}}!} \tag{7}$$

By defining the q-deformed exponential e_q by

$$\mathrm{e}_q : x \mapsto \mathrm{e}_q(x) := \sum_{n=0}^{k-1} \frac{x^n}{[n]_q!}$$

we can rewrite Eqs. (6) and (7) as

$$(z|z) = \mathrm{e}_q(\bar{z}z), \qquad (\bar{z}|\bar{z}) = \mathrm{e}_{\bar{q}}(z\bar{z})$$

(Observe that the summation in the exponential e_q is finite, for k finite, rather than infinite as is usually the case in q-deformed exponentials.)

We guess that the k-fermionic coherent states $|z)$ and $|\bar{z})$ form overcomplete sets with respect to some integration process accompanying the derivation process

inherent to Eq. (3). Following Majid and Rodríguez-Plaza [5], we consider the integration process defined by

$$\int dz\, z^p = \int d\bar{z}\, \bar{z}^p := 0 \quad \text{for} \quad p = 0, 1, \cdots, k-2 \tag{8a}$$

and

$$\int dz\, z^{k-1} = \int d\bar{z}\, \bar{z}^{k-1} := 1 \tag{8b}$$

Clearly, the integrals in (8) generalize the Berezin integrals corresponding to $k = 2$. In the case where k is arbitrary, we can derive the overcompleteness property

$$\int dz\, |z)\, \mu(z, \bar{z})\, (z|\, d\bar{z} = \int d\bar{z}\, |\bar{z})\, \mu(\bar{z}, z)\, (\bar{z}|\, dz = 1$$

where the function μ defined through

$$\mu(z, \bar{z}) := \sum_{n=0}^{k-1} ([n_q]![n_{\bar{q}}]!)^{\frac{1}{2}}\, z^{k-1-n}\, \bar{z}^{k-1-n}$$

may be regarded as a measure.

4 Fractional super-coherent states

We now switch to Q-deformed coherent states of the type

$$|Z) := \sum_{n=0}^{\infty} \frac{Z^n}{([n]_Q!)^{\frac{1}{2}}}\, |n\rangle \tag{9}$$

associated to a quon algebra A_Q where $Q \in \mathbf{C} \setminus S^1$. The latter states are simple deformations of the bosonic coherent states (cf. Ref. [10]). The coherent state $|Z)$ may be considered to be an eigenstate, with the eigenvalue $Z \in \mathbf{C}$, of an annihilation operator b_- in a representation such that the operator b_- and the associated creation operator b_+ satisfy

$$b_-|n\rangle = \left([n]_Q\right)^{\frac{1}{2}} |n-1\rangle \quad \text{with} \quad b_-|0\rangle = 0$$

$$b_+|n\rangle = \left([n+1]_Q\right)^{\frac{1}{2}} |n+1\rangle$$

with $n \in \mathbf{N}$.

For $Q \to q$, we have $[k]_Q! \to 0$. Therefore, the term $Z^k/([k]_Q!)^{\frac{1}{2}}$ in Eq. (9) makes sense for $Q \to q$ only if $Z \to z$, where z is a generalized Grassmann variable with $z^k = 0$. This type of reasoning has been invoked for the first time in Ref. [11]. (In [11], the authors show that there is an isomorphism between the braided line and the one-dimensional super-space.)

It is the aim of this section to determine the limit

$$|\xi) := \lim_{Q\to q}\lim_{Z\to z} |Z)$$

when Q goes to the root of unity $q = \exp(2\pi\mathrm{i}/k)$ and Z to a Grassmann variable z. The starting point is to rewrite Eq. (9) as

$$|Z) = \sum_{r=0}^{\infty}\sum_{s=0}^{k-1} \frac{Z^{rk+s}}{([rk+s]_Q!)^{\frac{1}{2}}} |rk+s\rangle$$

Then, by making use of the formulas

$$\frac{[k]_Q}{[rk]_Q} \to \frac{1}{r} \quad \text{for} \quad Q \to q \quad \text{with} \quad r \neq 0$$

and

$$\frac{[s]_Q}{[rk+s]_Q} \to 1 \quad \text{for} \quad Q \to q \quad \text{with} \quad s = 0, 1, \cdots, k-1$$

we find that

$$\lim_{Q\to q}\lim_{Z\to z} \frac{Z^{rk+s}}{([rk+s]_Q!)^{\frac{1}{2}}} = \frac{z^s}{([s_q]!)^{\frac{1}{2}}}\,\frac{\alpha^r}{(r!)^{\frac{1}{2}}} \tag{10}$$

works for $s = 0, 1, \cdots, k-1$ and $r \in \mathbf{N}$. The complex variable α in Eq. (10) is defined by

$$\alpha := \lim_{Q\to q}\lim_{Z\to z} \frac{Z^k}{([k]_Q!)^{\frac{1}{2}}}$$

Therefore, we obtain

$$|\xi) = \sum_{r=0}^{\infty}\sum_{s=0}^{k-1} \frac{z^s}{([s]_q!)^{\frac{1}{2}}}\,\frac{\alpha^r}{(r!)^{\frac{1}{2}}} |rk+s\rangle$$

Finally, by employing the symbolic notation

$$|rk+s\rangle \equiv |r\rangle \otimes |s\rangle$$

we arrive at the formal expression

$$|\xi) = \sum_{r=0}^{\infty} \frac{\alpha^r}{(r!)^{\frac{1}{2}}} |r\rangle \bigotimes \sum_{s=0}^{k-1} \frac{z^s}{([s]_q!)^{\frac{1}{2}}} |s\rangle \tag{11}$$

We thus end up with the product of a bosonic coherent state by a k-fermionic coherent state. This product shall be called a fractional super-coherent state. In the particular case $k = 2$, it reduces to the product of a bosonic coherent state by a fermionic coherent state, i.e., to the super-coherent state associated to a super-oscillator [12]. In the framework of field theory, Eq. (11) means that in the limit $Q \to q$, every field ψ with values $\psi(Z)$ is transformed into a fractional super-field Ψ with value $\Psi(z, \alpha)$, z being a generalized Grassmann variable and α a bosonic variable.

5 Concluding remarks

As a main result, the k-fermions introduced in the present paper can be ranged between fermions (for $k = 2$) and bosons (for $k \to \infty$). This result is further emphasized by calculating the coherence factor $g^{(m)}$ for an assembly of k-fermions: We find that $g^{(m)} = 0$ for $m > k-1$ so that, in a many-particle scheme, a given state of fractional spin $S = \frac{1}{k}$ cannot be occupied by more than $k-1$ identical k-fermions. The k-fermions thus satisfy a generalized Pauli exclusion principle.

We close this paper by mentioning two open questions. First, does the W_∞ algebra described by Fairlie, Fletcher and Zachos [13] plays an important role in the symmetries inherent to k-fermions (see also Ref. [14]) ? Second, what is the connection between k-fermions and fractional super-symmetry for anyons [15,16], especially the anyons constructed from unitary representations of the group diffeomorphisms of the plane [16] ? These matters should be the object of future works.

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